Inverted Oscillator

C. Yuce and A. Kilic
Physics Department, Anadolu University, Eskisehir, Turkey[*]

A. Coruh
Physics Department, Sakarya University, Sakarya, Turkey

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The inverted harmonic oscillator problem is investigated quantum mechanically. The exact wave function for the confined inverted oscillator is obtained and it is shown that the associated energy eigenvalues are discrete and it is given as a linear function of the quantum number n.

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I. INTRODUCTION

The exactly solvable potentials in quantum physics are very limited. The harmonic oscillator problem is one of the rare example of exactly solvable systems. Besides the mathematical interest, the non-perturbative exact treatment plays a crucial role in understanding and constructing for a new theory. For example, the harmonic oscillator problem is used in high energy physics and the state with zero-point energy is reinterpreted as the vacuum.
If the frequency ω is replaced with $i\omega$, the harmonic oscillator Hamiltonian becomes

$$H = -\frac{\partial^2}{\partial x^2} - \omega^2 x^2 \ , \tag{1}$$

where the constants are set to unity for simplicity ($\frac{\hbar^2}{2m} = \hbar = 1$). The potential in the new hamiltonian is known as the inverted harmonic oscillator potential or parabolic potential barrier. The inverted oscillator with an exponentially increasing mass is known as Caldriola-Kanai oscillator [1]. The inverted oscillator is the simplest system whose solutions to Newton equations diverge exponentially in phase space, a characteristic of chaotic motion.
Note that the replacement $\omega \to i\omega$ can not be applied to find the energy eigenvalues. If it can be made, the energy eigenvalues would take complex values for the Hermitian Hamiltonian (1). The reason not to apply it for finding the energy spectrum is due to the new boundary condition, namely the wave function doesn't vanish at infinity under such a replacement.

The inverted harmonic oscillator problem (1) attracts great attention not only for being one of the exactly solvable potential in quantum mechanics [2, 3, 4, 5, 6, 7, 8, 9] but also having wide range of application in many branches of physics. It receives a record number of applications in many branches of physics varying from high energy physics to solid state theory. For example, it is remarkable that 2-d string theory can be mapped on to the problem of non-interacting fermions in the inverted harmonic potential. All the physics of 2-d string theory can be recovered from this fermion theory. The massless tachyon field of string theory is related to small fluctuations around the Fermi surface [10]. Furthermore, the quantum sates of D0-brane decay are precisely the quantum states of the Hamiltonian (1) except that the spectrum starts at Fermi level and the model provides us with a complete description of the dynamics of a single D0-brane decay [11].
The using of inverted harmonic oscillator in the context of inflationary models was addressed by Guth and Pi [12]. In that work, the authors used the inverted harmonic oscillator as a toy model to describe the early time evolution of the inflation, starting from a Gaussian quantum state centered on the maximum of the potential.
Additionally, the parabolic potential with negative coupling (1) is used to understand the thermal activation problem. The analogy between the case of thermal activation and the case of the one-loop effective potential for theories with spontaneous symmetry breaking at tree level involve gaussian approximations around unstable configurations [13]. It was also shown in [13] that the problem of analyzing the decay of a metastable state becomes effectively that of a quantum mechanical inverted oscillator.
The inverted harmonic oscillator problem is also used as a model of instability [14, 15]. In the study of chaotic system,

[*]Electronic address: cyuce@anadolu.edu.tr

ω^2 is the instability parameter and determines the unstable and stable directions the rate at which initial phase space distributions expand and contract in these direction, respectively [14].
Another application of equation (1) is the statistical fluctuations of fission dynamics which is studied by means of the inverted oscillator [16].

The inverted harmonic oscillator is exactly solvable like standard harmonic oscillator which plays an important role in constructing the modern theories of physics. However, the inverted oscillator has a continuous energy spectrum and there is no zero-point energy associated with the inverted oscillator. It was also shown by many authors that the energy eigenstates are no longer square integrable.

In this study, we show that there is a direct link between the free particle and inverted harmonic oscillator problems. We solve the equation (1) analytically and it is shown that the inverted oscillator admits discrete energy spectrum. Furthermore, it is also shown that the wave function with discrete energy levels is square integrable but there is no zero-point energy contrary to standard harmonic oscillator.

II. FORMALISM

Let us now begin our study by introducing the following transformation for the wave function

$$\Psi(x,t) = \exp\left(\frac{i\omega x^2}{2} - \omega t\right)\Phi(x,t)\ . \tag{2}$$

By substituting the equation (2) into the Schrodinger equation with the Hamiltonian (1), we get the transformed equation.

$$-\frac{\partial^2\Phi}{\partial x^2} - 2i\omega x\ \frac{\partial\Phi}{\partial x} = i\frac{\partial\Phi}{\partial t}\ . \tag{3}$$

A canonical transformation is introduced as follows

$$x = q(t)\ y\ , \tag{4}$$

where $q(t)$ satisfies the classical equation of motion for the inverted harmonic oscillator potential. By solving the corresponding Lagrange equation with the potential $-\omega^2x^2$, the equation of motion is obtained

$$\frac{\ddot{q}(t)}{2} = 2\omega^2 q(t)\ . \tag{5}$$

Since the mass was set $(2m = 1)$, $1/2$ factor is introduced in front of $\ddot{q}$ instead of writing mass directly. Then, the solution follows

$$q(t) = e^{2\omega t}\ . \tag{6}$$

The canonical transformation (4) rescales the coordinate. The scaling parameter is given by solution of the corresponding Lagrange equation. It is just the classical path.
Under the transformation (4), the time derivative operator transforms as

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - 2\omega y\frac{\partial}{\partial y}\ . \tag{7}$$

If we substitute the equations (4,7) into the equation (3), we obtain

$$-\frac{\partial^2\Phi}{\partial y^2} = ie^{4\omega t}\ \frac{\partial\Phi}{\partial t}\ . \tag{8}$$

We can rewrite the above equation just by using the separation of variables technique

$$\Phi(y,t) = \exp\left(\ -\frac{i\epsilon}{4\omega}e^{-4\omega t}\ \right)\Phi(y)\ . \tag{9}$$

where ϵ is a constant and $\Phi(y)$ is given by

$$-\frac{\partial^2\Phi(y)}{\partial y^2} = -\epsilon\ \Phi(y)\ . \tag{10}$$

It is very interesting to observe that the new equation is just the Schrodinger equation for the free particle. A canonical transformation (4) connects the two Hamiltonians. So we conclude that the dynamical properties of the inverted harmonic oscillator and free particle can be inferred from each other.
By using the well-known wave functions for the free particle, we can obtain the wave functions for the inverted harmonic oscillator. There are two solutions of the free particle: the plane wave solution and the solution for which the particle is confined into a box. So, the two solutions for the inverted harmonic oscillator can be constructed as follows by using the equations (2,4,9)

$$\Psi_1(x,t) = N \exp\left(\frac{i\omega x^2}{2} - \omega t \ - \frac{i\epsilon}{4\omega} e^{-4\omega t}\right) \sin\left(e^{-2\omega t}\sqrt{\epsilon}\ x\right) \ , \tag{11}$$

$$\Psi_2(x,t) = \exp\left(\frac{i\omega x^2}{2} + ik\ e^{-2\omega t}\ x - \omega t \ + \frac{ik^2}{4\omega} e^{-4\omega t}\right) \ , \tag{12}$$

where N is a normalization constant and a new constant k for Ψ_2 is given by $k^2 = -\epsilon$. There is no normalization constant for Ψ_2, since the plane wave solution from which Ψ_2 is constructed can not be normalized.
The Hamiltonian (1) is invariant under the parity operator. It can be seen that Ψ_1 is an eigenket of the parity operator. It has an odd-parity. This is not the case for the standard harmonic oscillator, since the hermite polynomials H_n have the odd-parity for the odd-numbers of n and the even-parity for the even-numbers of n.
One another interesting case is the application of the time reversal operator ($t \to -t, i \to -i$). We see from the equations (11,12) that time reversal operation is equivalent to the following replacement: $\omega \to -\omega$. Fortunately, the Hamiltonian (1) is invariant under such a replacement. So, the wave functions under the application of the time reversal operator are also the solution.
Now, let us interpret the first solution (11) from the physical point of view. The sinusoidal character of Ψ_1 plays an important role since the confinement of the inverted oscillator can be achieved by this function. The particle can be confined in an expanding box. The stationary boundary condition is transformed to the moving boundary condition by the equation (4). The wave function is zero at both the origin and $x = L_0 e^{2\omega t}$, where L_0 is the initial length. So, the constant ϵ in (11) is given by ($\sqrt{\epsilon} = \dfrac{n\pi}{L_0}$). Note that this constant doesn't coincide the energy eigenvalues exactly, as can be seen below.
An important observation is made for the time-dependent function $q(t) = e^{2\omega t}$. It is just the particle position for the inverted oscillator in classical mechanics. However, in quantum mechanics, it is the position of the wall surrounding the particle. In other words, it is the boundary condition. The speed of the particle in classic physics and the the speed of the wall are exactly the same. In classical mechanics, the particle is exactly at the position $q(t) = e^{2\omega t}$, however in quantum mechanics, it may be found in the interval between the origin and the position $q(t) = e^{2\omega t}$.
Since the wall is expanding, one may say that the transition between the states may occur during the expansion. But, this is not the case, since the wave functions Ψ_1 for different values of n are orthogonal to each other. No transition occurs during the expansion of the wall.
Now, let us normalize the wave function Ψ_1 and then find it's the energy eigenvalues.

$$\int_0^{(L_0 e^{2\omega t})} |\Psi_1|^2 dx = |N|^2 \frac{L}{2} = 1 \ . \tag{13}$$

It is interesting to observe that the the normalized wave function can be found for the inverted harmonic oscillator in contrary to the existing idea. However, the second solution (12) is not normalized. The second solution is found to decay with the decay rate 4ω

$$|\Psi_2|^2 = e^{-2\omega t} = e^{-\frac{\Gamma}{2} t} \ , \tag{14}$$

where Γ is the decay rate. Having obtained the normalization constant, we can now compute the energy eigenvalues for Ψ_1. It is given by

$$E_n = \int_0^{(L_0 e^{2\omega t})} \Psi_1 H \Psi_1 dx = e^{-4\omega t} \frac{n^2 \pi^2}{L_0^2} \ . \tag{15}$$

The energy eigenvalues are decreasing as time goes on. The inverted oscillator admits discrete energy levels and it has a strong resemblance to free particle's energy spectrum. At the initial time, the energy eigenvalues for the inverted oscillator coincide with those of the free particle.As a special case, if ω is set to zero, then energy eigenvalues are

reduced to those of the free particle as it is expected.
There is no zero energy associated with the inverted oscillator in contrary to the standard harmonic oscillator. The energy depends on the quantum number linearly for the standard oscillator and quadratically for the confined inverted oscillator.
As an application, consider a quantum gas contained in an inverted harmonic oscillator potential. Let the quantum gas be confined in a box. One can investigate the problem statistically. Now, assume that the length (volume in 3D) of the box is increased exponentially. Then, the energy of each individual atom is decreased (15). During the expansion, no transition between the states in the statistical system occurs as it was explained before. Since the temperature of the system is related to the energy, we can safely say that the temperature is decreased when the volume is increased. The quantum gas gets cooler when the box expands. The dependence of temperature on volume for the inverted oscillator can be derived by using the relation (15) and the laws of statistical physics. The model studied here may be used for the non-equilibrium statistical physics.

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